A Problem of Relative, Constrained Motion

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Abstract

We develop a new method to determine the relative acceleration of a block sliding down along the face of a moving wedge. We have been able to link the solution of this problem to that of the inclined problem of elementary physics, thus providing a simpler solution to it.

Key words: Newton's Laws, Constrained Systems, Relative Motion, Friction, Relative Acceleration.

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The problem of determining the relative motion of a block sliding down on the surface of a wedge which is itself free to move along a frictionless horizontal plane [1, 2], as shown in *Fig. 1*, can be resolved by relating it to two problems of elementary Physics having well-known solutions.

First suppose that the wedge is held fast and consider the problem of a mass sliding down along the wedge's surface. This problem is equivalent to the elementary problem of a mass sliding down a frictionless inclined plane which makes an angle θ with the horizontal. The magnitude of the acceleration a_m of the mass moving down along the wedge's surface is thus ([3], p. 191)

$$a_m = g \sin\theta \tag{0.1}$$

Thus, when the acceleration of the wedge $a_M = 0$, the block acquires $a_m = g\sin\theta$ relative to the wedge.

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A slightly more sophisticated variant of the situation described above occurs when one wants to find out what acceleration should be imparted to the wedge in order to keep the block from sliding down along the wedge's surface. In this case, the normal force of reaction *(N)* of the wedge's surface on the block has a horizontal component which makes the block move along with the wedge with acceleration a_M. Thus $N \sin\theta = m a_M$. On the other hand, since there is no acceleration in the vertical direction, we have $N \cos\theta = mg$. Eliminating N we obtain

$$a_M = g \tan\theta \tag{0.2}$$

Thus, the block stays at rest ($a_m = 0$) relative to the wedge when $a_M = g \tan\theta$.

The reader will note ([3], p. 501) that when a simple pendulum is suspended from the roof of a car moving with acceleration a_M the string hangs at an angle from the vertical which is given by (0.2). The two solutions $(a_m, a_M) = (g \sin\theta, 0)$ and $(a_m, a_M) = (0, g \tan\theta)$ provide an easy way to determine relationship between the acceleration of the wedge and the acceleration of the block relative to the wedge for any value of the acceleration imparted to the wedge, from *zero* to $g \tan\theta$, where these extreme values correspond to the solutions of the two limiting cases discussed above. This is so because the variation of a_m is directly proportional to the normal force of reaction N, which, in turn, is also directly proportional to a_M. Thus, the relationship between the accelerations is a linear one, the pair of values (a_m, a_M) given above are two points on a straight line as shown in *Fig. 2*.

From this figure we get the general relationship between the accelerations

$$a_m = g \sin\theta - a_M \cos\theta. \tag{0.3}$$

Our point is that Eq. (0.3) also holds when the wedge moves solely under the weight of the sliding block, without any external force imparting an acceleration to the wedge. In this case, the linear momentum of the system (block and wedge) along the horizontal direction is conserved, that is:

$$p = (M + m)v_M + m v_m \cos\theta \tag{0.4}$$

where v_M is the velocity of the wedge relative to the floor and v_m is the velocity of the block relative to the wedge.

Eq. (0.4) implies this second relationship between the acceleration:

$$(M + m)a_M + m a_m \cos\theta = 0, \tag{0.5}$$

which is easily derived from the geometry of the system as in [4] and [5].

From (0.3) and (0.5) we solve the problem completely for a_M and a_m [1]:

$$a_M = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{0.6}$$

and

$$a_m = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta} \tag{0.7}$$

Even when there is friction between the block and the wedge an analogous relationship between the accelerations can be easily obtained by the same reasoning we have developed before, with minor changes to take friction into account.

Let μ be the coefficient of friction between the block and the wedge. Suppose also that $\mu < \tan\theta$. When $a_M = 0$ (the wedge is held fast again) the block acquires an acceleration ([7], p.72)

$$a_m = g(\sin\theta - \mu\cos\theta) \tag{0.8}$$

along the surface of the wedge.

If block and wedge do not move relative to each other, differently from the previous analysis, we have to consider two cases, according to whether the block is on the brink of moving upward or downward along the wedge's surface. We consider this latter situation, for which the balance of forces is $N(\sin\theta - \mu\cos\theta) = ma_M$ in the horizontal direction, and $N(\cos\theta + \mu\sin\theta) = mg$ in the vertical direction. Thus the block does not slide ($a_m = 0$) if the wedge's acceleration is [6]

$$a_M = g\frac{\sin\theta - \mu\cos\theta}{\sin\theta + \mu\cos\theta}. \tag{0.9}$$

The corresponding relationship between accelerations is again a linear one, as shown in *Fig. 3*, from which we deduce that

$$a_m = g(\sin\theta - \mu\cos\theta) - a_M(\cos\theta + \mu\sin\theta). \tag{0.10}$$

Note that Eq. (0.5) holds when friction is present as well. So from (0.5) and (0.10) we obtain (See [7] and numerous examples on p. 86, 87 for a physical insight into the meaning of these solutions)

$$a_M = -\frac{mg\cos^2\theta(\tan\theta - \mu)}{M + m - m\cos^2\theta(1 + \mu\tan\theta)} \tag{0.11}$$

and

$$a_m = \frac{(M+m)g\cos\theta(\tan\theta-\mu)}{M+m-m\cos^2\theta(1+\mu\tan\theta)}. \tag{0.12}$$

We leave it to the reader to justify our considering the situation in which the block is on the verge of sliding downward along the wedge instead of upward.

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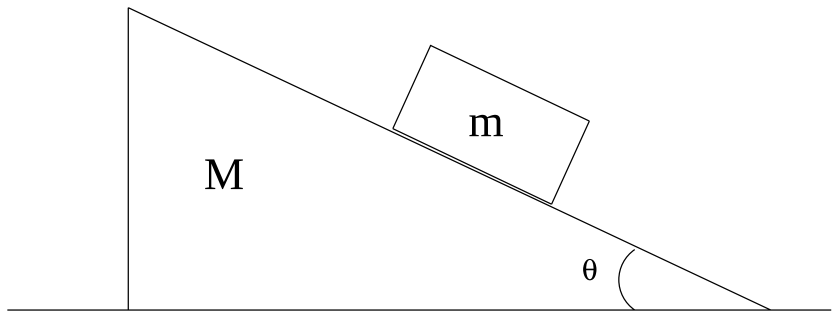

Figure 1: Block sliding on the wedge.

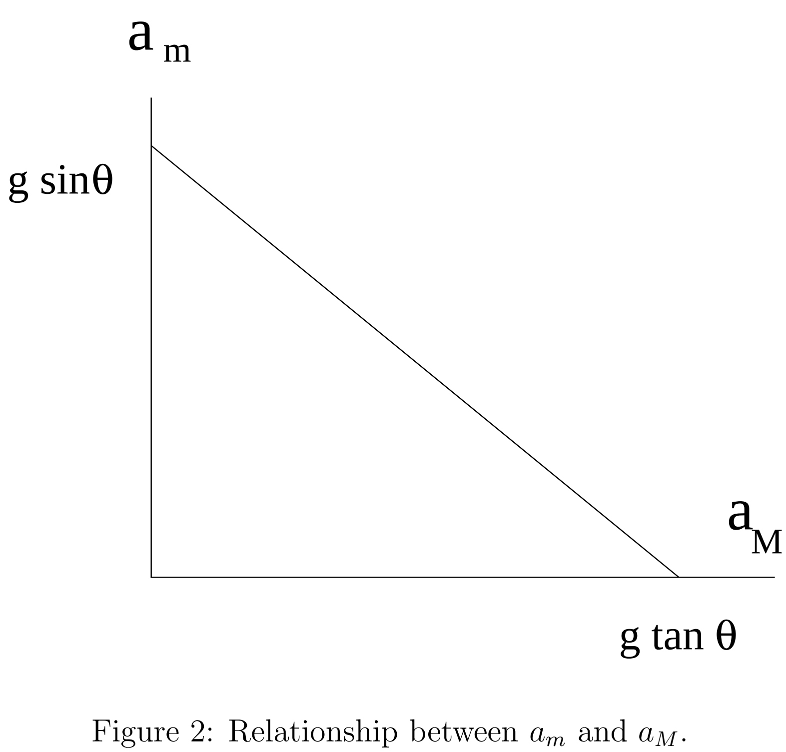

Figure 2: Relationship between a_m and a_M.

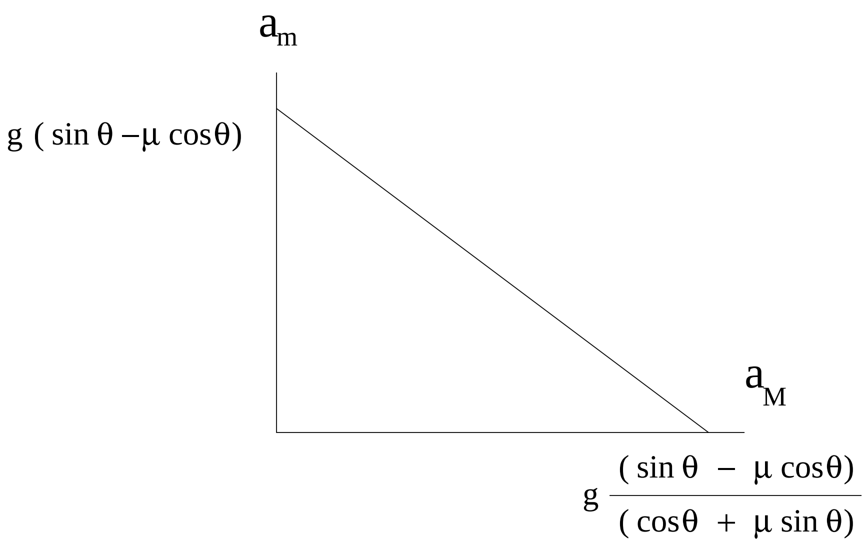

Figure 3: Relationship between a_m and a_M when $\mu \neq 0$.